Appendix E

Proofs for Section 3.2.4

Here we present the technical details for Section 3.2.4.

Def. Given $A \in U$, define $MAX(A) = \{a \in A \mid \forall b \in A. \neg(a < b)\}$. That is, $MAX(A)$ consists of the maximal elements of A.

Zorn's Lemma. Let P be a non-empty ordered set in which every chain has an upper bound. Then P has a maximal element.

Prop. E.1. $\forall A \in U.\ A \subseteq {\downarrow}MAX(A)$, and hence $A = {\downarrow}MAX(A)$.

Proof. Pick $A \in U$ and $a \in A$ and define $P_a = \{x \in A \mid a \leq x\}$. For all chains $C \subseteq P_a$, C is a directed set and $C \subseteq A$, so $b = \bigvee C \in A$ (since A is closed). If C is not empty, then $a \leq b$ so $b \in P_a$. Thus, every chain in P_a has an upper bound in P_a, so by Zorn's Lemma, P_a has a maximal element d. If there is any $c \in A$ such that $d < c$ then $a < c$ so $c \in P_a$, contradicting the maximality of d in P_a. Thus $d \in MAX(A)$ and $a \in {\downarrow}MAX(A)$. Therefore $A \subseteq {\downarrow}MAX(A)$. Clearly $MAX(A) \subseteq A$, and, since A is closed, ${\downarrow}MAX(A) \subseteq {\downarrow}A \subseteq A$ and so $A = {\downarrow}MAX(A)$. ■

Prop. E.2. $\forall A, B \in U.\ A = B \Leftrightarrow MAX(A) = MAX(B)$.

Proof. Assume A and B are in U. Clearly, $A = B \Rightarrow MAX(A) = MAX(B)$. To show the converse, assume $A \neq B$ and, without loss of generality, that $a \in A$ & $a \notin B$. Since $A \subseteq {\downarrow}MAX(A)$, there must be $c \in MAX(A)$ with $a \leq c$. However, since B is a down-set,

$c \notin B$, and hence $c \notin MAX(B)$. Thus $MAX(A) \neq MAX(B)$. ■

Prop. E.3. $\forall A \in U.\ A \equiv_R MAX(A)$.

Proof. First, $MAX(A) \leq_R A$, since $MAX(A) \subseteq A$. Now, if $A \cap C \neq \phi$ for $C \subseteq X$ open then $\exists a \in A \cap C$. Now, $A \subseteq \downarrow MAX(A)$ so $\exists b \in MAX(A).\ a \leq b$. However, since C is open $b \in C$ so $b \in A \cap C$ and $MAX(A) \cap C \neq \phi$. Thus $A \leq_R$ MAX(A) and $A \equiv_R MAX(A)$. ■

Prop. E.4. Given a tuple type $t = struct\{t_1;...;t_n\} \in T$, $A \in F_t$ and $a = a_1 \vee ... \vee a_n \in A$, where $\forall i\ .\ a_i \in A_i \in F_{t_i}$, then $a \in MAX(A) \Leftrightarrow \forall i.\ a_i \in MAX(A_i)$.

Proof. Note that a and the a_i are tuples, and the *sup* of tuples is taken componentwise, so $\forall s \in S.\ a_s = a_{1s} \vee ... \vee a_{ns}$. Also note that $i \neq j \Rightarrow SC(t_i) \cap SC(t_j) = \phi$. If there is some i such that $a_i \notin MAX(A_i)$, then $\exists b_i \in A_i.\ a_i < b_i$ so $b = a_1 \vee ... \vee b_i \vee ... \vee a_n \in A$. Now, $a_i < b_i \Rightarrow \exists s \in S.\ a_{is} < b_{is}$ and (since $j \neq i \Rightarrow a_{js} = \perp = b_{js}$) $a_s = a_{is}$ and $b_s = b_{is}$, so $a < b$. Thus $a \notin MAX(A)$. Conversely, if $a \notin MAX(A)$ then $\exists b \in A.\ a < b$ with $a = a_1 \vee ... \vee a_n$, $b = b_1 \vee ... \vee b_n$, and $\forall i.\ a_i, b_i \in A_i$. For some $s \in S$, $a_s < b_s$. Thus $b_s > \perp$ so $\exists j.\ s \in SC(t_j)$, and so $a_s < b_s \Rightarrow a_j < b_j$ (since $a_s = a_{js}$ and $b_s = b_{js}$). Thus $a_j \notin MAX(A_j)$. ■

Prop. E.5. For all types $t \in T$ and all $A \in F_t$, $MAX(A)$ is finite. If $t \in S$ and $A = \downarrow(\perp,...,a,...,\perp) \in F_t$ then $MAX(A) = \{(\perp,...,a,...,\perp)\}$. If $t = struct\{t_1;...;t_n\} \in T$ and $A = \{(a_1 \vee ... \vee a_n) \mid \forall i.\ a_i \in A_i\} \in F_t$ then $MAX(A) = \{(a_1 \vee ... \vee a_n) \mid \forall i.\ a_i \in MAX(A_i)\}$. If $t = (array\ [w]\ of\ r) \in T$ and $A = \{a_1 \vee a_2 \mid g \in G\ \&\ a_1 \in E_w(g)\ \&\ a_2 \in E_r(a(g))\} \in F_t$ then $MAX(A) = \{a_1 \vee a_2 \mid g \in G\ \&\ a_1 \in MAX(E_w(g))\ \&\ a_2 \in MAX(E_r(a(g)))\}$.

Proof. We will demonstrate this proposition by induction on the structure of t. Let $t \in S$ and let $A \in F_t$. Then $\exists a \in I_s.\ A = \downarrow(\bot,...,a,...,\bot)$, so $MAX(A) = \{(\bot,...,a,...,\bot)\}$. $MAX(A)$ has a single member and is thus finite.

Let $t = struct\{t_1;...;t_n\} \in T$ and let $A \in F_t$. By Prop. E.4, $MAX(A) = \{(a_1 \vee ... \vee a_n) \mid \forall i.\ a_i \in MAX(A_i)\}$. By the inductive hypothesis, the $MAX(A_i)$ are finite, so $MAX(A)$ is finite.

Let $t = (array\ [w]\ of\ r) \in T$ and let $A \in F_t$. There is a finite set $G \in FIN(H_w)$ and a function $a \in (G \to H_r)$ such that

$$A = \{a_1 \vee a_2 \mid g \in G\ \&\ a_1 \in E_w(g)\ \&\ a_2 \in E_r(a(g))\} =$$
$$\bigcup\{\{a_1 \vee a_2 \mid a_1 \in E_w(g)\ \&\ a_2 \in E_r(a(g))\} \mid g \in G\} = \bigcup\{A_g \mid g \in G\}$$

where we define $A_g = \{a_1 \vee a_2 \mid a_1 \in E_w(g)\ \&\ a_2 \in E_r(a(g))\}$. Each A_g is an object in $F_{struct\{w;\, r\}}$ for the tuple type $struct\{w;\ r\}$. By Prop. E.4,

$$MAX(A_g) = \{a_1 \vee a_2 \mid a_1 \in MAX(E_w(g))\ \&\ a_2 \in MAX(E_r(a(g)))\} =$$
$$\{(\bot,...,g,...,\bot) \vee a_2 \mid a_2 \in MAX(E_r(a(g)))\}$$

Pick $g \neq g'$ in G, and $b \in MAX(A_g)$ and $b' \in MAX(A_{g'})$. Then there are $b_2 \in MAX(E_r(a(g)))$ and $b_2' \in MAX(E_r(a(g')))$ such that $b = (\bot,...,g,...,\bot) \vee b_2$ and $b' = (\bot,...,g',...,\bot) \vee b_2'$. If $b > b'$ then $g > g'$ since $b_{2w} = b_{2w}' = \bot$. However, this contradicts the defintion of $FIN(H_w)$. Thus no $b \in MAX(A_g)$ is larger than any $b' \in MAX(A_{g'})$ for $g \neq g'$ in G. Thus

$$MAX(A) = MAX(\bigcup\{A_g \mid g\in G\}) = \bigcup\{MAX(A_g) \mid g\in G\} =$$
$$\bigcup\{\{a_1 \vee a_2 \mid a_1 \in MAX(E_w(g)) \;\&\; a_2 \in MAX(E_r(a(g)))\} \mid g\in G\} =$$
$$\{a_1 \vee a_2 \mid g\in G \;\&\; a_1 \in MAX(E_w(g)) \;\&\; a_2 \in MAX(E_r(a(g)))\}.$$

G is finite, and by the inductive hypothesis, $MAX(E_w(g))$ and $MAX(E_r(a(g)))$ are finite, so $MAX(A)$ is finite. ■

Appendix F

Proofs for Section 3.4.1

Here we present the technical details for Section 3.4.1. First, two definitions are given to provide the context for the work in this and subsequent appendices.

Def. Let S denote a finite set of scalars, let $X = \mathbf{X}\{I_s \mid s \in S\}$ denote a set of tuples, and let $U = CL(X)$ denote the lattice of data objects consisting of closed sets of tuples whose primitive values are taken from the scalars in S.

Def. Let DS denote a finite set of display scalars, let $Y = \mathbf{X}\{I_d \mid d \in DS\}$ denote a set of tuples, and let $V = CL(Y)$ denote the lattice of displays consisting of closed sets of tuples whose primitive values are taken from the display scalars in DS.

Now we prove four propositions that we will use as lemmas in other proofs.

Prop. F.1. For all $A, B \in U$, $\downarrow A \wedge \downarrow B = \downarrow(A \wedge B)$.

Proof. $\downarrow A \wedge \downarrow B = \downarrow A \cap \downarrow B = \{C \mid C \leq A\} \cap \{C \mid C \leq B\} = \{C \mid C \leq A \;\&\; C \leq B\} = \{C \mid C \leq A \wedge B\} = \downarrow(A \wedge B)$. ■

Prop. F.2. $D(\phi) = \phi$ and $D(\{(\bot,...,\bot)\}) = \{(\bot,...,\bot)\}$.

Proof. First, note that $\forall u \in U.\ \phi \leq u$ and $\forall u \in U.\ u \neq \phi \Rightarrow \{(\bot,...,\bot)\} \leq u$. That is, ϕ is the least element in U, and $\{(\bot,...,\bot)\}$ is the next largest element in U. If

$D(\phi) = v > \phi$, then $\exists u \in U.\ D(u) = \phi$ and $u < \phi$, which is impossible. Thus $D(\phi) = \phi$. Similarly, if $D(\{(\perp,...,\perp)\}) = v > \{(\perp,...,\perp)\}$, then $\exists u \in U.\ D(u) = \{(\perp,...,\perp)\}$ and $u < \{(\perp,...,\perp)\}$. However, the only $u < \{(\perp,...,\perp)\}$ is ϕ, and $D(\phi) = \phi$, so $D(\{(\perp,...,\perp)\}) = \{(\perp,...,\perp)\}$. ■

Prop. F.3. If $D{:}U \rightarrow V$ is a display function, then its inverse D^{-1} is a continuous function from $D(U)$ to U.

Proof. First, D^{-1} is a function since D is injective, and D^{-1} is monotone since D is an order embedding. D^{-1} is continuous if for all directed $M \subseteq D(U)$, $\bigvee D^{-1}(M) = D^{-1}(\bigvee M)$. However, since D is a homomorphism, $D^{-1}(M)$ is a directed set in U. Thus, since D is continuous, $\bigvee D(D^{-1}(M)) = D(\bigvee D^{-1}(M))$, and so $D^{-1}(\bigvee D(D^{-1}(M))) = D^{-1}(D(\bigvee D^{-1}(M)))$. This simplifies to $D^{-1}(\bigvee M) = \bigvee D^{-1}(M)$, showing that D^{-1} is continuous. ■

Prop. F.4. If $D{:}U \rightarrow V$ is a display function, then $\forall M \subseteq D(U).\ \bigvee D^{-1}(M) = D^{-1}(\bigvee M)$.

Proof. Given $M \subseteq D(U)$ let $N = D^{-1}(M) \subseteq U$. By Prop. B.2, $\bigvee D(N) = D(\bigvee N)$, which is equivalent to $\bigvee M = D(\bigvee D^{-1}(M))$, and applying D^{-1} to both sides of this, we get $D^{-1}(\bigvee M) = D^{-1}(D(\bigvee D^{-1}(M))) = \bigvee D^{-1}(M)$. ■

Now we define an open neighborhood of a tuple in X, and prove two more lemmas. Note that in the following we will use the notation a_S to indicate the s component of a tuple $a \in \mathsf{X}\{I_S \mid s \in S\}$.

Def. Given a tuple $a \in \mathsf{X}\{I_S \mid s \in S\}$ such that $a_S \neq [x, x]$ for continuous s, define *neighbor*(a) as the set of tuples b such that:

s discrete $\Rightarrow b_s \geq a_s$

s continuous and $a_s = \perp \Rightarrow b_s \geq a_s$

s continuous and $a_s \neq \perp \Rightarrow b_s > a_s$

(that is $a_s = [x, y]$ and $b_s = [u, v] \Rightarrow x < u$ and $v < y$).

Prop. F.5. For $a \in \mathsf{X}\{I_s \mid s \in S\}$, the set *neighbor*($a$) is open (in the Scott topology).

Proof. Clearly *neighbor*(a) is an up set. Let C be a directed set in $\mathsf{X}\{I_s \mid s \in S\}$ such that $d = \bigvee C$ belongs to *neighbor*(a). The *sup* is taken componentwise, so $d_s = \bigvee\{c_s \mid c \in C\}$ for each s. If s is discrete, then $\exists c^s \in C.\ c^s{}_s = d_s > a_s$. If s is continuous and $a_s = \perp$, then for any $c \in C$, $c_s \geq a_s$. If s is continuous and $a_s \neq \perp$, then a_s and d_s are intervals such that $d_s = [u, v] \subset [x, y] = a_s$, with $x < u$ and $v < y$. Here $u = \max\{p \mid \exists c \in C.\ [p, q] = c_s\}$ and $v = \min\{q \mid \exists c \in C.\ [p, q] = c_s\}$ so there exist $c^s{}_1$, $c^s{}_2 \in C$ such that $c^s{}_{1s} = [p_1, q_1]$ and $c^s{}_{2s} = [p_2, q_2]$ with $x < p_1$ and $q_2 < y$. Since C is directed, there must be $c^s \in C$ such that $c^s \geq c^s{}_1 \vee c^s{}_2$, so $c^s{}_s > a_s$. For each $s \in S$ we have shown that there is $c^s \in C$ such that $c^s{}_s \geq a_s$. Since S is finite, and C is directed, there is $c \in C$ such that $c \geq \bigvee\{c^s \mid s \in S\} \geq a$ and $c \in$ *neighbor*(a). Thus *neighbor*(a) is an open set. ■

Prop. F.6. Given a set $C \subseteq U$, $B = \bigvee C$ and an open set A in $\mathsf{X}\{I_s \mid s \in S\}$, then $A \cap B \neq \phi \Rightarrow \exists c \in C.\ A \cap c \neq \phi$.

Proof. B and all $c \in C$ are closed, so B is the smallest closed set containing $\bigcup C$. All the $c \in C$ are down sets, so $\bigcup C$ is also a down set. Thus, by Prop. C.10,

$\{\bigvee M \mid M \subseteq \bigcup C \ \&\ M$ directed$\}$ is closed and hence equal to B. We are given that there is a $y \in A \cap B$, so there must be a directed set M in $\bigcup C$ such that $y = \bigvee M$. However, since A is open, there must be $m \in M \cap A$, and since $M \subseteq \bigcup C$, there is $c \in C$ such that $m \in c \cap A$. ■

Now we define the embeddings of scalar objects and display scalar objects in the lattices U and V.

Def. For each scalar $s \in S$, define an embedding $E_s{:}I_s \rightarrow U$ by: $\forall b \in I_s.\ E_s(b) = \downarrow(\bot,...,b,...,\bot)$ (this notation indicates that all elements of the tuple are $\bot$ except b). Also define $U_s = E_s(I_s) \subseteq U$.

Def. For each display scalar $d \in DS$, define an embedding $E_d{:}I_d \rightarrow V$ by: $\forall b \in I_d.\ E_d(b) = \downarrow(\bot,...,b,...,\bot)$. Also define $V_d = E_d(I_d) \subseteq V$.

Next, we use an argument involving open neighborhoods to show that a display function maps embedded scalar objects to displays of the form $\downarrow x$, where x is a display tuple. Prop. F.8 will show that these $\downarrow x$ must be embedded display scalar objects.

Prop. F.7. If $D{:}U \rightarrow V$ is a display function, then for all $s \in S$, $\forall b \in I_s.\ \exists x \in \mathsf{X}\{I_d \mid d \in DS\}.\ D(\downarrow(\bot,...,b,...,\bot)) = \downarrow x$.

Proof. Given $s \in S$ and $b \in I_s$, let $a = (\bot,...,b,...,\bot)$ and let $z = D(\downarrow a)$. Then $z = \bigvee\{\downarrow y \mid y \in z\}$, and by Prop. F.4, $\downarrow a = D^{-1}(z) = \bigvee\{D^{-1}(\downarrow y) \mid y \in z\}$ (note $\downarrow y \leq z$ so $D^{-1}(\downarrow y)$ exists).

Now we know that $a \in \bigvee\{D^{-1}(\downarrow y) \mid y \in z\}$. If we could show that $\bigvee\{D^{-1}(\downarrow y) \mid y \in z\} = \bigcup\{D^{-1}(\downarrow y) \mid y \in z\}$ then there must be $x \in z$ such that $a \in D^{-1}(\downarrow x)$. However, the $D^{-1}(\downarrow y)$ are closed sets, and, by Prop. C.8, we can only show that $\bigvee\{D^{-1}(\downarrow y) \mid y \in z\} = \bigcup\{D^{-1}(\downarrow y) \mid y \in z\}$ if z is finite. Thus we need a more complex argument to construct $x \in z$ such that $a \in D^{-1}(\downarrow x)$.

Define a sequence of tuples a_n in U, for n =1, 2, ..., by:

if s is continuous and $b = a_s = [x, y]$ for some interval $[x, y]$, then

$a_{ns} = [x\text{-}1/n, y\text{+}1/n]$

if s is continuous and $b = a_s = \perp$, then $a_{ns} = \perp$

if s is discrete, then $a_{ns} = a_s$

for all $s' \in S$ such that $s' \neq s$, $a_{ns'} = \perp$

Also define $z_n = D(\downarrow a_n) \leq D(\downarrow a) = z$, and note that $\downarrow a_n = \bigvee\{D^{-1}(\downarrow x) \mid x \in z_n\}$. Now $neighbor(a_{n-1})$ is open and $\downarrow a_n \cap neighbor(a_{n-1}) \neq \phi$, so by Prop. F.6 there must be $x_n \in z_n$ such that $D^{-1}(\downarrow x_n) \cap neighbor(a_{n-1}) \neq \phi$. Say y is in this intersection. Then $y \in neighbor(a_{n-1}) \Rightarrow a_{n-1} \leq y$ and $y \in D^{-1}(\downarrow x_n) \Rightarrow \downarrow y \leq D^{-1}(\downarrow x_n)$ so $\downarrow a_{n-1} \leq \downarrow y \leq D^{-1}(\downarrow x_n)$. Furthermore, $x_n \in z_n \Rightarrow D^{-1}(\downarrow x_n) \leq D^{-1}(z_n) = \downarrow a_n$, so we have $\downarrow a_{n-1} \leq D^{-1}(\downarrow x_n) \leq \downarrow a_n$, or equivalently $\downarrow x_{n-1} \leq D(\downarrow a_{n-1}) \leq \downarrow x_n$. Thus $x_{n-1} \leq x_n$ and the set $\{x_n\}$ is a chain and thus a directed set. Since $\mathbf{X}\{I_d \mid d \in DS\}$ is a *cpo*, $x = \bigvee\{x_n\} \in \mathbf{X}\{I_d \mid d \in DS\}$. Since $z \in U$, z is a closed under sups and thus $x \in z$.

Now, $\forall n.\ x_n \leq x$ so $\forall n.\ \downarrow a_n \leq D^{-1}(\downarrow x_{n+1}) \leq D^{-1}(\downarrow x)$. Thus $\downarrow a = \bigvee_n \downarrow a_n \leq D^{-1}(\downarrow x)$ (note that $a \in D^{-1}(\downarrow x)$) and $D(\downarrow a) \leq \downarrow x$. On the other hand, $x \in z \Rightarrow \downarrow x \leq z = D(\downarrow a)$, and so $D(\downarrow a) = \downarrow x$. ■

Prop. F.7 showed that a display function maps embedded scalar objects to displays of the form $\downarrow x$, where x is a display tuple. Now we show that these $\downarrow x$ must be embedded display scalar objects, and that embedded scalar objects are mapped to embedded display scalar objects of the same kind (that is, discrete or continuous).

Prop. F.8. If $D{:}U \to V$ is a display function, then

$\forall s \in S.\ \forall a \in U_S.\ \exists d \in DS.\ D(a) \in V_d$.

Furthermore, if s is discrete, then d is discrete, and if s is continuous, then d is continuous.

Proof. A value $u \in U_S$ has the form $u = \downarrow(\perp,...,a,...,\perp)$. If $a = \perp$ then $D(u) = \{(\perp,...,\perp)\}$ which belongs to V_d for all $d \in DS$. Otherwise, by Prop. F.7, $\exists v \in \mathbf{X}\{I_d \mid d \in DS\}.\ D(u) = \downarrow v$ and by Prop. F.2, $\downarrow v > \{(\perp,...,\perp)\}$. If $\downarrow v$ is not in any V_d, then some $(...,e,...,f,...) \in \downarrow v$ with $e \neq \perp \neq f$. We consider the discrete and continuous cases separately.

First, consider s discrete. We have $\downarrow(...,e,...,\perp,...) < \downarrow v$ and $\exists u' \in U$ such that $D(u') = \downarrow(...,e,...,\perp,...) < \downarrow v = D(u)$, so $u' < u$. But the only u' less than u are ϕ and $\{(\perp,...,\perp)\}$, and D does not carry them into $\downarrow(...,e,...,\perp,...)$. Thus $\downarrow v$ must be in some V_d.

Second, consider s continuous. Define $w_{ef} = (\perp,...,e,...,f,...,\perp)$ (that is, e and f are the only elements in this tuple that are not $\perp$). Also define $v_e = \downarrow(\perp,...,e,...,\perp,...,\perp)$ and $v_f = \downarrow(\perp,...,\perp,...,f,...,\perp)$. Then $v_e, v_f < \downarrow w_{ef} \leq \downarrow v = D(u)$ so $\exists u_e, u_f < u.\ (D(u_e) = v_e \ \&\ D(u_f) = v_f)$. Now, $v_e \neq \{(\perp,...,\perp)\}$ so $u_e \neq \{(\perp,...,\perp)\}$ and $\exists a_e \neq \perp.\ (\perp,...,a_e,...,\perp) \in u_e$ and hence $\downarrow(\perp,...,a_e,...,\perp) \leq u_e$. Similarly, $\exists a_f \neq \perp.\ \downarrow(\perp,...,a_f,...,\perp) \leq u_f$. By Prop. F.1, $\downarrow(\perp,...,a_e \wedge a_f,...,\perp) \leq u_e \wedge u_f$. However, a_e and a_f are real intervals (since they belong to a continuous scalar and are not $\perp$), so $a_e \wedge a_f$ is the smallest interval containing both a_e and a_f. Let a_g be this interval. Then

$a_g = a_e \wedge a_f \neq \bot$, and $\downarrow(\bot,...,a_g,...,\bot) \leq u_e \wedge u_f$. Thus $u_e \wedge u_f \neq \{(\bot,...,\bot)\}$. On the other hand, $v_e \wedge v_f = \{(\bot,...,\bot)\}$. But this contradicts $D(u_e \wedge u_f) = v_e \wedge v_f$, so $\downarrow v$ must be in some V_d.

Next we show that discrete scalar values map to discrete scalar values, and that continuous scalar values map to continuous scalar values.

Let $u = \downarrow(\bot,...,a,...,\bot) \in U_s$ for discrete s with $D(u) = v = \downarrow(\bot,...,b,...,\bot) \in V_d$ and $b \neq \bot$. If d is continuous, then $\exists b'. \bot < b' < b$ such that $\{(\bot,...,\bot)\} < \downarrow(\bot,...,b',...,\bot) = v' < v$. Thus $\exists u'. D(u') = v'$ where $\{(\bot,...,\bot)\} < u' < u = \downarrow(\bot,...,a,...,\bot)$. Thus $u' = \downarrow(\bot,...,a',...,\bot)$ where $a' < a$, which is impossible for discrete s, so d must be discrete.

Let $u = \downarrow(\bot,...,a,...,\bot) \in U_s$ for continuous s with $D(u) = v = \downarrow(\bot,...,b,...,\bot) \in V_d$. Then $\exists a'. \bot < a' < a$ and $\{(\bot,...,\bot)\} < \downarrow(\bot,...,a',...,\bot) = u' < u$, so $D(\{(\bot,...,\bot)\}) = \{(\bot,...,\bot)\} < D(u') = v' < v$. This is only possible if V_d is continuous. ■

Next we show that embedded objects from different scalars are not mapped to the same display scalar embedding.

Prop. F.9. If $D{:}U \to V$ is a display function, then for all s and s' in S, $(s \neq s' \;\&\; u_a \in U_s \;\&\; u_b \in U_{s'} \;\&\; u_a \neq \bot \neq u_b \;\&\; D(u_a) \in V_d \;\&\; D(u_b) \in V_{d'}) \Rightarrow d \neq d'$.

Proof. Let $v_a = D(u_a)$ and $v_b = D(u_b)$. Assume that v_a and v_b are in the same V_d, and let $u_a = \downarrow(\bot,...,a,...,\bot,...,\bot)$,

$u_b = \downarrow(\bot,...,\bot,...,b,...,\bot)$,

$v_a = \downarrow(\bot,...,e,...,\bot)$ and

$v_b = \downarrow(\bot,...,f,...,\bot)$, where $a \neq \bot \neq b$ and $e \neq \bot \neq f$.

This notation indicates that u_a and u_b are in different U_s, and that v_a and v_b are in the same V_d.

First, we treat the continuous case. $u_a \wedge u_b = \{(\bot,...,\bot)\}$ and, by Prop. F.1, $v_a \wedge v_b = \downarrow(\bot,...,e \wedge f,...,\bot)$. e and f are real intervals, and $e \wedge f$ is the smallest interval containing both e and f. Thus $e \wedge f \neq \bot$ so $v_a \wedge v_b \neq \{(\bot,...,\bot)\}$, which contradicts $D(u_a \wedge u_b) = v_a \wedge v_b$. Thus v_a and v_b must be in the same V_d.

Second, treat the discrete case. Note that
$u_a \vee u_b = \{(\bot,...,a,...,\bot,...,\bot), (\bot,...,\bot,...,b,...,\bot), (\bot,...,\bot)\}$ and
$D(u_a \vee u_b) = v_a \vee v_b = \{(\bot,...,e,...,\bot), (\bot,...,f,...,\bot), (\bot,...,\bot)\}$.
Let $x = \downarrow(\bot,...,a,...,b,...,\bot) =$
$\{(\bot,...,a,...,b,...,\bot), (\bot,...,a,...,\bot,...,\bot), (\bot,...,\bot,...,b,...,\bot), (\bot,...,\bot)\} > u_a \vee u_b$.
Set $y = D(x)$. Then $y > v_a \vee v_b$ so there is $(\bot,...,g,...,\bot) \in y$ (all elements of this tuple are $\bot$ except g) such that $(\bot,...,e,...,\bot) \neq (\bot,...,g,...,\bot) \neq (\bot,...,f,...,\bot)$. [In fact $(\bot,...,g,...,\bot)$ may not even be in the same V_d that $(\bot,...,e,...,\bot)$ and $(\bot,...,f,...,\bot)$ are in.] Now if $\downarrow(\bot,...,g,...,\bot) = y$ then $e \leq g$ and $f \leq g$ which is impossible in the discrete order of I_d. Thus $\downarrow(\bot,...,g,...,\bot) < y$ and so $\exists w < x.\ D(w) = \downarrow(\bot,...,g,...,\bot)$. However, the only w less than x are ϕ, $\{(\bot,...,\bot)\}$, u_a , u_b and $u_a \vee u_b$. This contradicts $g \neq e$ and $g \neq f$. Thus v_a and v_b must be in the same V_d. ■

As a corollary of Prop. F.9, we show that only embedded scalar objects are mapped to embedded display scalar objects (that is, non-scalar objects must be mapped to non-display scalar objects).

Prop. F.10. If $D{:}U \rightarrow V$ is a display function, then
$\forall d \in DS.\ (D(u) \in V_d \Rightarrow \exists s \in S.\ u \in U_s)$.

Proof. If $u \in U$ is not in any scalar embedding, then $\exists(\ldots,e,\ldots,f,\ldots) \in u.\ e \neq \bot \neq f$. Assume $D(u) = v \in V_d$. Then $(\bot,\ldots,e,\ldots,\bot,\ldots,\bot) \in u$ and $(\bot,\ldots,\bot,\ldots,f,\ldots,\bot) \in u$, so $\downarrow(\bot,\ldots,e,\ldots,\bot,\ldots,\bot) \leq u$ and $\downarrow(\bot,\ldots,\bot,\ldots,f,\ldots,\bot) \leq u$, and thus $D(\downarrow(\bot,\ldots,e,\ldots,\bot,\ldots,\bot)) \in V_d$ and $D(\downarrow(\bot,\ldots,\bot,\ldots,f,\ldots,\bot)) \in V_d$. However $\downarrow(\bot,\ldots,e,\ldots,\bot,\ldots,\bot)$ and $\downarrow(\bot,\ldots,\bot,\ldots,f,\ldots,\bot)$ are in two different scalar embeddings and, by Prop. F.9, cannot both be mapped to V_d. Thus $D(u)$ cannot belong to any display scalar embedding. ■

Next, we show that all embedded objects from a continuous scalar are mapped to embedded objects from the same display scalar. Note, however, that embedded objects from the same discrete scalar may be mapped to embedded objects from different display scalars.

Prop. F.11. If $D{:}U \rightarrow V$ is a display function and if s is a continuous scalar, then $\forall u_a, u_b \in U_s.\ ((D(u_a) \in V_d \ \&\ D(u_b) \in V_{d'} \ \&\ u_a \neq \bot \neq u_b) \Rightarrow d = d')$.

Proof. Let $v_a = \mathrm{D}(u_a)$ and $v_b = \mathrm{D}(u_b)$. Assume that s is continuous and that v_a and v_b are in different V_d. Let

$u_a = \downarrow(\bot,\ldots,a,\ldots,\bot)$,

$u_b = \downarrow(\bot,\ldots,b,\ldots,\bot)$,

$v_a = \downarrow(\bot,\ldots,e,\ldots,\bot,\ldots,\bot)$ and

$v_b = \downarrow(\bot,\ldots,\bot,\ldots,f,\ldots,\bot)$, where $a \neq \bot \neq b$ and $e \neq \bot \neq f$.

This notation indicates that u_a and u_b are in the same U_s, and that v_a and v_b are in different V_d. Now $v_a \wedge v_b = \{(\bot,\ldots,\bot)\}$ and, by Prop. F.1, $u_a \wedge u_b = \downarrow(\bot,\ldots,a \wedge b,\ldots,\bot)$. Since a and b are real intervals, $a \wedge b$ is the smallest interval containing both a and b, so $a \wedge b \neq \bot$. However, this contradicts $D(u_a \wedge u_b) = v_a \wedge v_b$. Thus, v_a and v_b must be in the same V_d. ■

Now we show that a display function maps objects of the form $\downarrow a$, for $a \in \mathbf{X}\{I_s \mid s \in S\}$, to objects of the form $\downarrow x$, for $x \in \mathbf{X}\{I_d \mid d \in DS\}$, and conversely. Furthermore, the values of display functions on objects of the form $\downarrow a$ are determined by their values on embedded scalar objects. Given this, it is an easy step in Prop. F.13 to show that the values of display functions on all of U are determined by their values on embedded scalar objects.

Prop. F.12. If $D:U \to V$ is a display function and if a is a tuple in $\mathbf{X}\{I_s \mid s \in S\}$ then there exists a tuple x in $\mathbf{X}\{I_d \mid d \in DS\}$ such that $D(\downarrow a) = \downarrow x$. Conversely, if x is a tuple in $\mathbf{X}\{I_d \mid d \in DS\}$ such that $\exists A \in U.\ x \in D(A)$, then there exists a tuple a in $\mathbf{X}\{I_s \mid s \in S\}$ such that $D(\downarrow a) = \downarrow x$. From Prop. F.8 we know that for all $s \in S$,

$a_s \neq \bot \Rightarrow \exists d \in DS.\ \exists y_d \in I_d.\ (y_d \neq \bot\ \&\ \downarrow(\bot,\ldots,y_d,\ldots,\bot) = D(\downarrow(\bot,\ldots,a_s,\ldots,\bot)))$,

and similarly, from Prop. D.3 we know that for all $d \in DS$,

$x_d \neq \bot \Rightarrow \exists s \in S.\ \exists b_s \in I_s.\ (b_s \neq \bot\ \&\ \downarrow(\bot,\ldots,x_d,\ldots,\bot) = D(\downarrow(\bot,\ldots,b_s,\ldots,\bot)))$,

Here we assert that for all $s \in S$, $a_s \neq \bot \Rightarrow a_s = b_s$, and for all $d \in DS$, $x_d \neq \bot \Rightarrow x_d = y_d$. That is, the tuple elements of a determine the tuple elements of x, and vice versa, according to the values of D on the scalar embeddings U_s.

Proof. This is similar to the proof of Prop F.7. Given $a \in \mathbf{X}\{I_s \mid s \in S\}$, let $z = D(\downarrow a)$. Then $z = \bigvee\{\downarrow y \mid y \in z\}$, and by Prop. F.4, $\downarrow a = D^{-1}(z) = \bigvee\{D^{-1}(\downarrow y) \mid y \in z\}$ (note $\downarrow y \leq z$ so $D^{-1}(\downarrow y)$ exists).

Define a sequence of tuples a_n in U, for n =1, 2, ..., by:

s discrete $\Rightarrow a_{ns} = a_s$

s continuous and $a_s = \perp \Rightarrow a_{ns} = a_s$

s continuous and $a_s = [x, y] \Rightarrow a_{ns} = [x\text{-}1/n, y\text{+}1/n]$.

Also define $z_n = D(\downarrow a_n) \leq D(\downarrow a) = z$, and note that $\downarrow a_n = \bigvee\{D^{-1}(\downarrow x) \mid x \in z_n\}$. Now $neighbor(a_{n-1})$ is open and $\downarrow a_n \cap neighbor(a_{n-1}) \neq \phi$. By Prop. F.6 there must be $x_n \in z_n$ such that $D^{-1}(\downarrow x_n) \cap neighbor(a_{n-1}) \neq \phi$. Say y is in this intersection. Then $y \in neighbor(a_{n-1}) \Rightarrow a_{n-1} \leq y$ and $y \in D^{-1}(\downarrow x_n) \Rightarrow \downarrow y \leq D^{-1}(\downarrow x_n)$ so $\downarrow a_{n-1} \leq \downarrow y \leq D^{-1}(\downarrow x_n)$. Furthermore, $x_n \in z_n \Rightarrow D^{-1}(\downarrow x_n) \leq D^{-1}(\downarrow z_n) = \downarrow a_n$, so we have $\downarrow a_{n-1} \leq D^{-1}(\downarrow x_n) \leq \downarrow a_n$.

Now consider the tuple components of a_n and x_n. Define x_n' by $\downarrow(\perp,\ldots,x_{nd}',\ldots,\perp) = D(\downarrow(\perp,\ldots,a_{ns},\ldots,\perp))$, and set $x_{nd}' = \perp$ for those d not corresponding to any $a_{ns} \neq \perp$. Also define a_n' by $\downarrow(\perp,\ldots,x_{nd},\ldots,\perp) = D(\downarrow(\perp,\ldots,a_{ns}',\ldots,\perp))$ for those d such that $x_{nd} \neq \perp$, and set $a_{ns}' = \perp$ for those s not corresponding to any $x_{nd} \neq \perp$. Note that $\downarrow(\perp,\ldots,x_{nd},\ldots,\perp) \leq \downarrow x_n$ so $\exists w \in U.\ \downarrow(\perp,\ldots,x_{nd},\ldots,\perp) = D(w)$, and, by Prop. D.3, w must have the form $\downarrow(\perp,\ldots,a_{ns}',\ldots,\perp)$, so a_{ns}' exists for $x_{nd} \neq \perp$. First, we use $D^{-1}(\downarrow x_n) \leq \downarrow a_n$ to show that:

(a) $\downarrow(\perp,\ldots,x_{nd},\ldots,\perp) \leq \downarrow x_n \Rightarrow$

$\downarrow(\perp,\ldots,a_{ns}',\ldots,\perp) = D^{-1}(\downarrow(\perp,\ldots,x_{nd},\ldots,\perp)) \leq D^{-1}(\downarrow x_n) \leq \downarrow a_n \Rightarrow$

$a_{ns}' \leq a_{ns} \Rightarrow$

$\downarrow(\perp,\ldots,a_{ns}',\ldots,\perp) \leq \downarrow(\perp,\ldots,a_{ns},\ldots,\perp) \Rightarrow$

$\downarrow(\perp,\ldots,x_{nd},\ldots,\perp) = D(\downarrow(\perp,\ldots,a_{ns}',\ldots,\perp)) \leq$

$D(\downarrow(\perp,\ldots,a_{ns},\ldots,\perp)) = \downarrow(\perp,\ldots,x_{nd}',\ldots,\perp) \Rightarrow$

$x_{nd} \leq x_{nd}'$

The transition from the fourth to the fifth line in (a) shows that if a_{ns} and a_{ns}' are in the same scalar s, then x_{nd} and x_{nd}' are in the same display scalar d. Next, we use $\downarrow a_n \le D^{-1}(\downarrow x_{n+1})$ to show that:

(b) $\downarrow(\bot,...,a_{ns},...,\bot) \le \downarrow a_n \Rightarrow$

$\downarrow(\bot,...,x_{nd}',...,\bot) = D(\downarrow(\bot,...,a_{ns},...,\bot)) \le D(\downarrow a_n) \le \downarrow x_{n+1} \Rightarrow$

$x_{nd}' \le x_{(n+1)d} \Rightarrow$

$\downarrow(\bot,...,x_{nd}',...,\bot) \le \downarrow(\bot,...,x_{(n+1)d},...,\bot) \Rightarrow$

$\downarrow(\bot,...,a_{ns},...,\bot) = D^{-1}(\downarrow(\bot,...,x_{nd}',...,\bot)) \le$

$D^{-1}(\downarrow(\bot,...,x_{(n+1)d},...,\bot)) = \downarrow(\bot,...,a_{(n+1)s}',...,\bot) \Rightarrow$

$a_{ns} \le a_{(n+1)s}'$

The transition from the fourth to the fifth line in (b) shows that if x_{nd} and $x_{(n+1)d}'$ are in the same display scalar d, then a_{ns} and $a_{(n+1)s}'$ are in the same scalar s.

Putting (a) and (b) together shows that $a_{ns}' \le a_{ns} \le a_{(n+1)s}'$ and $x_{nd} \le x_{nd}' \le x_{(n+1)d}$ for all s and d. If d is a discrete display scalar, then there is an n such that $\forall m \ge n.\ x_{md} = x_{nd}$, and define $x_d = x_{nd}$. If d is a continuous display scalar, then there either all the x_{nd} are $\bot$ or there is an n such that $\forall i, j \ge n.\ i \ge j \Rightarrow x_{id} = [u_i, v_i] \subseteq [u_j, v_j] = x_{jd}$. In the first case, define $x_d = \bot$ and in the second case define $x_d = [u, v] = \bigcap\{[u_i, v_i] \mid i \ge n\}$. In any case, $x_d = \bigvee_n x_{nd}$, and defining x as the tuple with components x_d, $x = \bigvee_n x_n$. Since z is closed, $\{x_n\}$ is a directed set, and $\forall n.\ x_n \in z$, then $x \in z$.

By definition, $a = \bigvee_n a_n$. We have already shown that $\downarrow a_{n-1} \leq D^{-1}(\downarrow x_n)$, so $a_{n-1} \in D^{-1}(\downarrow x_n) \subseteq D^{-1}(\downarrow x)$. Since $D^{-1}(\downarrow x)$ is closed, $a \in D^{-1}(\downarrow x)$ and thus $\downarrow a \leq D^{-1}(\downarrow x)$. However, $x \in z$, so $D^{-1}(\downarrow x) \leq \downarrow a$ and thus $\downarrow a = D^{-1}(\downarrow x)$. Define x' and a' by $\downarrow(\perp,...,x_{nd}',...,\perp) = D(\downarrow(\perp,...,a_{ns},...,\perp))$ and $\downarrow(\perp,...,x_{nd},...,\perp) = D(\downarrow(\perp,...,a_{ns}',...,\perp))$. Then we can apply the logic of (a) and (b) (using $\downarrow a \leq D^{-1}(\downarrow x) \leq \downarrow a$) to show that $a_s' \leq a_s \leq a_s'$ and $x_d \leq x_d' \leq x_d$, which is just $a_s = a_s'$ and $x_d = x_d'$. Thus D takes the set of tuple components of a into exactly the set of tuple components of x.

For the converse, we are given a tuple x in $\mathbf{X}\{I_d \mid d \in DS\}$ such that $\exists A \in U.\ x \in D(A)$. Then $\downarrow x \leq D(A)$ and $\exists z \leq A.\ \downarrow x = D(z) = \bigvee\{D(\downarrow b) \mid b \in z\}$. After this, the argument for the converse is identical, relying on properties of D that are shared by D^{-1}. D^{-1} is a homomorphism from $D(U)$ to U, and Props. F.3 and F.4 show that D^{-1} is continuous and preserves arbitrary *sups*. In the argument D^{-1} is only applied to members of V that are less than $\downarrow x$, where D^{-1} is guaranteed to be defined. ■

Proposition F.13 will show that the values of display functions on all of U are determined by their values on the scalar embeddings U_S, which is particularly interesting since most elements of U cannot be expressed as *sups* of sets of elements of the scalar embeddings U_S.

Prop. F.13. If $D{:}U \rightarrow V$ is a display function, then its values on U are determined by its values on the scalar embeddings U_S.

Proof. For all $u \in U$, $u = \bigvee\{\downarrow x \mid x \in u\}$. By Prop. B.2, $D(u) = \bigvee\{D(\downarrow x) \mid x \in u\}$. Now, each $x \in u$ is a tuple so by Prop. F.12, $D(\downarrow x)$ is determined by the values of D applied to the tuple components of x. Thus $D(u)$ is determined by the values of D on the scalar embeddings U_S. ■

The propositions in Appendix F are combined in the following definition and theorem about mappings from scalars to display scalars.

Def. Given a display function D, define a mapping MAP_D: $S \rightarrow POWER(DS)$ by $MAP_D(s) = \{d \in DS \mid \exists a \in U_s.\ D(a) \in V_d\}$.

Theorem. F.14. Every display function D:$U \rightarrow V$ is an injective lattice homomorphism whose values are determined by its values on the scalar embeddings U_s. D maps values in the scalar embedding U_s to values in the display scalar embeddings V_d for $d \in MAP_D(s)$. Furthermore,

s discrete and $d \in MAP_D(s) \Rightarrow d$ discrete,

s continuous and $d \in MAP_D(s) \Rightarrow d$ continuous,

$s \neq s' \Rightarrow MAP_D(s) \cap MAP_D(s') = \phi$,

s continuous $\Rightarrow MAP_D(s)$ contains a single display scalar.