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AN EFFICIENT METHOD OF INTERPOLATING OBSERVATIONS TO UNIFORMLY SPACED GRIDS

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1. INTRODUCTION

Objective analysis of meteorological data sets can be accomplished using weighted averages to interpolate to a uniform rectangular grid. The most noted examples are the methods of Cressman (1959) and Barnes (1964) which are widely used today. A similar method which closely approximates the results of the Barnes method, but which allows for considerably faster computing time, is currently employed on the McIDAS at the Space Science and Engineering Center of the University of Wisconsin- Madison. Specifically, if we wish to interpolate NS observed data points to NG grid points, the Barnes and Cressman methods take computing time proportional to NS*NG. The new method takes time proportional to NS+NG, which runs about 30 times faster than the Barnes method on McIDAS when there are approximately 1000 observations and 1500 grid points.

2. INTERPOLATION BY WEIGHTING FUNCTIONS

The analysis of a variable (P) at a uniformly spaced grid of locations (x,y) is usually accomplished by forming a weighted sum of all data values S_i.

$$P(x,y) = \frac{\sum_{i=1}^{NS} W(x-x_i,\ y-y_i)*S_i}{\sum_{i=1}^{NS} W(x-x_i,\ y-y_i)} \tag{1}$$

where (i) denotes the ith point of the NS observations at the coordinates (x_i, y_i). The weighting function is usually inversely proportional to the distance from the grid point to the observation.

$$W(x,y) = \exp\,[-(x-x_i)^2/r - (y-y_i)^2/r] \tag{2}$$

This, in effect is a method of low pass filtering observations proposed by Barnes (1964). Higher frequency detail can be added back into the analysis by using the difference between the observation and the analysis value interpolated to the location of the observation.

$$T_i = S_i - P\,(x_i, y_i) \tag{3}$$

This difference is included through a second weight sum.

$$Q(x,y) = \frac{\sum_{i=1}^{NS} W(x-x_i, y-y_i)*T_i}{\sum_{i=1}^{NS} W(x-x_i, y-y_i)} \tag{4}$$

Note that the weight sums (1) and (4) are essentially convolutions of the observations (S_i) with weighting function W (x,y).

3. AN ALTERNATIVE METHOD

The calculations of the weighted sums (1) and (4) can be speeded up considerably if the weight functions are picked from a restricted class so it is not necessary to calculate the products of the weight and data for every pair of grid point and data point $(x-x_i, y-y_i)$. To do this, we take advantage of running summations and emulate the Barnes (1964) low pass filter technique through a combination of operations on the observations.

The method is easiest to illustrate in a one-dimensional case. Let x be the grid point location and x_i the observation locations (Fig. 1). The value at x can be determined from the

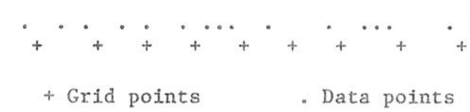


Figure 1: Schematic representation of a one-dimensional analysis

ratio of two sums, A(x) and B(x).

$$P(x) = A(x)/B(x) \tag{5}$$

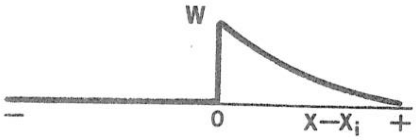
The sums A(x) and B(x) are calculated as:

$$A(x) = \sum_{i=1}^{NS} W(x-x_i)*S_i \tag{6}$$

$$B(x) = \sum_{i=1}^{NS} W(x-x_i) \tag{7}$$

The weighting function we used is a one-sided exponential

$$W(x-x_i) = \begin{cases} \exp\,(-(x-x_i)/r) & x_i \le x \\ 0 & x_i > x \end{cases} \quad (8)$$



where r is a constant that determines the width of the function or spatial smoothing of the data. Using this one-sided weighting function, the sums A(x) and B(x) can be factored into two sums

$$A(x)=\sum_{x_i\le x-\Delta x}\exp(-(x-x_i)/r)*S_i+\sum_{x-\Delta x<x_i\le x}\exp(-(x-x_i)/r)*S_i \quad (9)$$

The first sum represents all the data points prior to or to the left of the grid point previous to x, which is $x-\Delta x$ (Δx being a fixed distance between grid points). The second sum considers only the additional data between the last grid point, $x-\Delta x$, and the current point, x. This division into two sums is the key factor in increasing computation speed which will become more clearly understood later.

The expression $\exp(-(x-x_i)/r)$ can be factored into the product of two terms, $\exp(-\Delta x)/r)*\exp(-(x-\Delta x-x_i)/r)$, which allows (9) to become

$$A(x)=\exp(-\Delta x/r)*\sum_{x_i\le x-\Delta x}\exp(-(x-\Delta x-x_i)/r)*S_i+\sum_{x-\Delta x<x_i\le x}\exp(-(x-x_i)/r)*S_i$$

which is really a recurrence relationship for A.

$$A(x)=\exp(-\Delta x/r)*A(x-\Delta x)+\sum_{x-\Delta x<x_i\le x}\exp(-(x-x_i)/r)*S_i \quad (10)$$

This implies that A(x) can be computed from the past sum used for the last grid point $A(x-\Delta x)$ with the addition of the data points between $x-\Delta x$ and x, which is the second part of (10). This, by itself, is not the correct value for the grid point since it has to be normalized for the weighting functions used in 10. The denominator B(x) is similar to the numerator A(x)

$$B(x) = \sum_{x_i\le x}\exp\,(-(x-x_i)/r) \quad (11)$$

which has the recurrence relation

$$B(x) = \exp\,(-\Delta x/r) * B\,(x-\Delta x) + \sum_{x-\Delta x<x_i\le x}\exp\,(-(x-x_i)/r) \quad (12)$$

To compute P(x), the recurrence relationships allow the weight sums of the data to be calculated only once for each data point. An additional exponential has to be added for the interval between grid points. Thus, the exponential function is only called NS+NG times in the first pass.

To completely analyze P(x), additional passes through the grid are required. Since the first pass (left to right conceptually) used a one-sided weighting function, a complementary pass using a complementary weighting function is required. The second pass is in the reverse direction of the first (right to left), and the weighting function also is skewed the opposite of (8).

$$W(x-x_i) = \begin{cases} 0 & x_i < x \\ \exp\,((x-x_i)/r) & x_i \ge x \end{cases} \quad (13)$$

Then $A(x) = \sum_{x_i\ge x}\exp((x-x_i)/r)*S_i$ gives the recurrence relations.

$$A(x)=\exp(-\Delta x/r)*A(x+\Delta x)+\sum_{x+\Delta x\ge x_i>x}\exp((x-x_i)/r)*S_i \quad (14)$$

$$B(x)=\exp(-\Delta x/r)*B(x+\Delta x)+\sum_{x+\Delta x\ge x_i>x}\exp((x-x_i)/r) \quad (15)$$

To approximate a Barnes method more fully, two additional passes (left to right and complementary right to left) must be made through the data. These passes are necessary because the Barnes weight function (2) is an exponential of the <u>square</u> of the distance (x^2), while our method is only an exponential of the <u>linear</u> distance x, (8) and (13). The two pairs of passes through the data are added to approximate the Barnes weighting function (2). In order to approximate the Barnes weight function $\exp(-(x-x_i)^2/r^2)$, we use $r_1=r/2.34$ and $r_2=r/2.75$ to give the four weighting functions.

$$W_1(x-x_i)=\exp(-(x-x_i)/r_1),$$
$$W_2(x-x_i)=\exp(-(x-x_i)/r_2),$$
$$W_3(x-x_i)=\exp\,((x-x_i)/r_1),$$
and $W_4(x-x_i)=\exp\,((x-x_i)/r_2)$

which imply a total weight of

$$\exp(-(x-x_i)^2/r^2\cong 8.96*[W_1(x-x_i)+W_3(x-x_i)]-8*[W_2(x-x_i)+W_4(x-x_i)] \quad (16)$$

4. APPLICATION TO 2-DIMENSIONAL GRIDS

For a two-dimensional grid, the same procedure applies as described for the one-dimensional case except that we must now consider a two-dimensional weighting function

$$W(x-x_i,y-y_i) = \begin{cases} \exp(-(x-x_i)/t)*\exp(-(y-y_i)/q) & \text{for } x_i\le x \text{ and } y_i\le y \\ 0 & \text{for } x_i>x \text{ or } y_i>y \end{cases} \quad (17)$$

As in the one-dimensional case, the grid point value P(x) is derived from the ratio of the two sums.

$$P(x,y) = A(x,y)/B(x,y) \quad (18)$$

The sums are defined as follows:

$$A(x,y)=\sum_{\substack{x_i\le x\\ y_i\le y}}\exp(-(x-x_i)/t)*\exp(-(y-y_i)/q)*S_i \quad (19)$$

$$B(x,y) = \sum_{\substack{x_i \le x \\ y_i \le y}} \exp(-(x-x_i)/t) * \exp(-(y-y_i)/q) \qquad (20)$$

But to compute these sums in two dimensions, we must move both from left to right and top to bottom at the same time. This is done by defining an intermediate sum $A^1(x,y)$ as

$$A^1(x,y) = \sum_{\substack{x-\Delta x < x_i \le x \\ y_i \le y}} \exp(-(x-x_i)/t) * \exp(-(y-y_i)/q) * S_i \qquad (21)$$

Then $A^1(x,y)$ can be computed quickly over the two-dimensional grid using the recurrence relation defined in the y direction as

$$A^1(x,y) = \exp(-\Delta y/q) * A^1(x,y-\Delta y) + \sum_{\substack{x-\Delta x < x_i \le x \\ y-\Delta y < y_i \le y}} \exp(-(x-x_i)/t) * \exp(-(y-y_i)/q) * S_i \qquad (22)$$

Now $A(x,y)$ can be computed over the two-dimensional grid, using the recurrence relation defined in the x direction as

$$A(x,y) = \exp(-\Delta x/t) * A(x-\Delta x,y) + A^1(x,y) \qquad (23)$$

A similar relationship holds for the denominator of (18) with a $B^1(x,y)$ defined as

$$B^1(x,y) = \sum_{\substack{x-\Delta x < x_i \le x \\ y_i \le y}} \exp(-(x-x_i)/t) * \exp(-(y-y_i)/q) \qquad (24)$$

The sums (22) and (23) are calculated in a rastor scan from the upper left corner of the grid, moving left to right for (22) and top to bottom for (23). Additional passes have to be made through the grid from the upper right corner, working rastor scans from the bottom toward the top. Thus, four passes have to be made through the grid with one weighting function. The Barnes weighting function in two dimensions can be broken in the product of two one-dimensional exponential functions.

$$W(x-x_i, y-y_i) = \exp(-((x-x_i)^2 + (y-y_i)^2)/r) = \exp(-(x-x_i)^2/r) * \exp(-(y-y_i)^2/r) \qquad (25)$$

If we multiply out the approximations to these functions, as done in (16) which considers the reverse direction pases through the data, equation (25) will have 16 terms. Thus, for a two-dimensional analysis, 16 passes have to be made through the grid, but each pass requires the exponential to be calculated for only the sum of the NG grid points and NS data points. The Barnes method would still require the number of operations to be proportional to the product of the number of grid and data points (NS*NG).

The exponential functions can be approximated with finite internal lookup tables as an additional step for computing efficiency. This has often been used for the implementation of the Barnes scheme. However, in the algorithm presented here, the finite approximation to the exponential can be made with more precision because the exponential is restricted to a shorter distance than in the Barnes method. The largest distance an exponential will be applied to is Δx, the distance between grid points, since the weighted sums for data points more than Δx from a grid point are included through the recurrence relationships (14), (22), and (23).

The recurrence relationships have to be initialized at the first grid point. This is done by considering the data outside of the grid and applying the basic form of the weighted sums (19), (20), (21) and (24) that are not factored into the recurrence relationships. These sums then form the basis for applying the recurrence relationships to all succeeding grid points.

Distance calculations ($x-x_i$, $y-y_i$) are done in units of degrees latitude and longitude. Corrections for confluence of meridions at high latitude also are made to ($x-x_i$) interms of COS(Y). This allows the weighting functions to approximate the spherical distances between data and grid points.

5. CONCLUDING REMARKS

This method closely approximates the Barnes method where data are present (Figs. 2 and 3). The only situation where a deviation

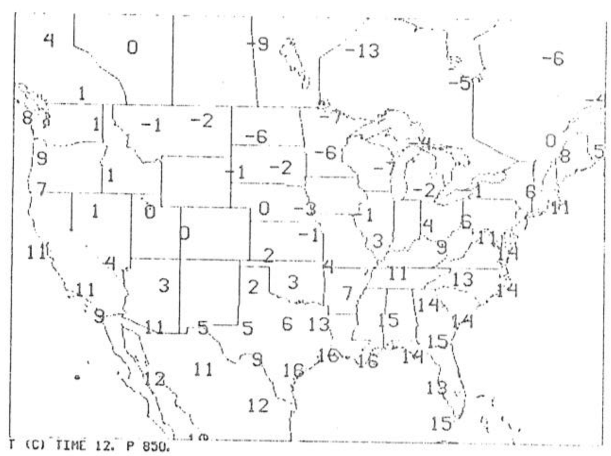


Figure 2: Example data set of radiosonde 850 mb temperatures.

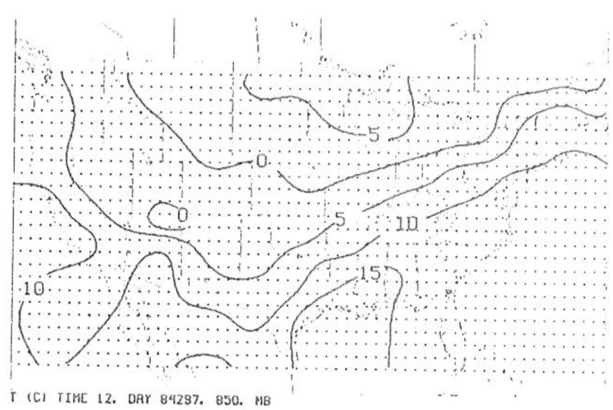


Figure 3: Resulting gridded analysis made of temperatures shown in Fig. 2. Grid point spacing is 1°.

from the Barnes method can be found is in large data void areas where information has to be extrapolated over long distances. The differences arise because we cannot perfectly approximate an exponential distance squared $(\exp(-(x-x_i)^2/r))$ with a sum of linear exponentials. This approximation is very close at short distances $(x-x_i)$, but deviates out at the tails where $x-x_i$ is large. Our method gives a slightly larger weight to the extrapolated data because of this approximation. However, in data void areas, there really is no information in the analysis since the physics of the variable are not considered. For this reason, our analysis method is considered to be as accurate as Barnes and Cressman and a valid substitute.

This method also can be generalized to any number of dimensions. The number of computations required will always be proportioned to NS+NG for any number of dimensions in the grid.

6. REFERENCES

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