

## ON EXACT CONVERGENCE OF THE ACCELERATED OVERRELAXATION METHOD WHEN APPLIED TO CONSISTENTLY ORDERED SYSTEMS

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The problem of determining the exact regions of convergence and divergence of the block Accelerated Overrelaxation (AOR) iterative method, when it applies to systems with a Generalized Consistently Ordered (GCO) coefficient matrix, is addressed here. Some new algebraic results in the theory of regular splittings are obtained and used for the determination of extended regions of convergence. Complementary, in some cases, divergence regions are obtained by making use of a recently derived eigenvalue functional equation.

**KEY WORDS:** Iterative methods, accelerated overrelaxation, generalized consistently ordered matrices, regular splittings, eigenvalues.

**CR CATEGORIES:** AMS(MOS): 65F10, 65F15

### 1. INTRODUCTION

During the last decade, a number of interesting convergence results, concerning the two-parametric AOR [1, 2] method, appeared in the literature (e.g. [3-24]). Most of these results deal with the fundamental problems of convergence and optimization of the AOR method, as it pertains to cyclic consistently ordered matrices. In particular:

i) for a class of nonsymmetric 2-cyclic consistently ordered matrices, arising from the discretization of elliptic PDEs by the Hermite-bicubic finite element collocation method, it is shown [15, 19, 22] that the optimal AOR is always faster than the optimal Successive Overrelaxation (SOR [25, 26]) method.

ii) for a class of 3-cyclic consistently ordered matrices, arising from the least-squares solution of large overdetermined systems, it is shown [20, 24] that the optimal 3-block AOR method is always faster than the corresponding optimal 3-block SOR method.

The above results establish the fact that the AOR method is a competitive scheme, and motivated us to further develop its underlying convergence theory.

Here, as a physical continuation of the work in [23], we deal with the problem of exactly determining the boundaries of the largest convergence domain (defined

by the AOR's two parameters) of the AOR method, as it pertains to a whole class of GCO matrices. Our main purposes are:

- i) To obtain extended convergence domains, using the theory of nonnegative matrices and regular splittings (Section 2).
- ii) To derive some upper bounds of convergence, using the eigenvalue functional equation of [23] (Section 3).
- iii) To determine the regions where the above complement each other (Section 4).

To fix notation, consider the system of linear equations

$$Ax = b, \quad (1.1)$$

where  $A$  is a nonsingular  $n \times n$  complex matrix ( $A \in \mathbb{C}^{n,n}$ ).

Writing  $A$  as

$$A = D(I - L - U), \quad (1.2)$$

where  $D$  is a nonsingular block diagonal matrix and  $L, U$  are respectively strictly lower and strictly upper triangular matrices, it is well known that the associated Jacobi iteration matrix  $B^A$  can be expressed as

$$B^A = L + U. \quad (1.3)$$

Using now the definition in [27], the matrix  $A$  of (1.2) is GCO  $(s, q)$ , if all the eigenvalues of the matrix  $B(a) = a^s L + a^{-q} U$  are independent of  $a$ ,  $a \neq 0$  ( $s$  and  $q$  are positive integers). In this case we also say that  $B^A$  of (1.3) is a GCO  $(s, q)$ -matrix.

For any matrix  $C = [c_{ij}]$  in  $\mathbb{C}^{n,n}$ , let  $|C|$  denote the matrix in  $\mathbb{R}^{n,n}$  with entries  $|c_{ij}|$ . Throughout this paper we assume that the matrix  $A$  of (1.2) belongs to the matrix set

$$\mathcal{G} := \{A \in \mathbb{C}^{n,n} / |B^A| = |L| + |U| \text{ is a GCO } (s, q)\text{-matrix}\}. \quad (1.4)$$

Apparently, if  $B^A$  of (1.3) is a nonnegative GCO  $(s, q)$ -matrix or can be permuted to a certain diagonal form (cf. [28–30]) then (cf. [27]) both  $B^A$  and  $|B^A|$ , and hence  $A$ , are GCO  $(s, q)$ -matrices.

The block AOR method, applied to the matrix equation (1.1) is, as usual, defined by (cf. [1])

$$\begin{cases} x^{(m+1)} = \mathcal{L}_{r,\omega} x^{(m)} + c_{r,\omega}, & m = 0, 1, 2, \dots \\ \mathcal{L}_{r,\omega} \equiv \mathcal{L}_{r,\omega}^A := (I - rL)^{-1} [(1 - \omega)I + (\omega - r)L + \omega U], \\ c_{r,\omega} := \omega(I - rL)^{-1} D^{-1} b \end{cases} \quad (1.5)$$

The parameters  $r$  and  $\omega \neq 0$  are respectively referred to as acceleration and overrelaxation factors. By their special combinations, well known iterative schemes are recovered. For instance, when  $(r, \omega) = (0, 1)$ ,  $(1, 1)$  and  $(\omega, \omega)$ , the AOR reduces to Jacobi, Gauss-Seidel and SOR methods, respectively. Finally, observe that upon writing

$$\mathcal{L}_{r,\omega} = I - \omega(I - rL)^{-1}D^{-1}A \quad \text{and} \quad c_{r,\omega} = (I - \mathcal{L}_{r,\omega})A^{-1}b \quad (1.6)$$

and since  $A$  is nonsingular and  $\omega \neq 0$ , the AOR method is completely consistent ([31], p. 64).

## 2. DOMAINS OF CONVERGENCE

In this Section, the theory of nonnegative matrices and regular splittings is used to obtain domains of convergence for the AOR method.

Given the matrix  $B \equiv B^A$  of (1.3), let  $\bar{\mu}$  denote the spectral radius of

$$|B| = |L| + |U| \equiv |B^A| \quad (2.1)$$

namely

$$\bar{\mu} := \rho(|B|). \quad (2.2)$$

Then, it is known (cf. [23]) that:

**LEMMA 2.1** *Let  $|B|$  of (2.1) be a GCO  $(s, q)$ -matrix and  $p := s + q$ . Then, for any real nonnegative constants  $\alpha$ ,  $\beta$  and  $\gamma$ , with  $\gamma \neq 0$ , satisfying*

$$\alpha^q \beta^s \bar{\mu}^p < \gamma^p, \quad (2.3)$$

*the matrix  $\bar{A} := \gamma I - \alpha|L| - \beta|U|$  is such that*

$$\bar{A}^{-1} \geq 0. \quad (2.4)$$

Now, let the matrices  $M$ ,  $N$ ,  $\tilde{M}$  and  $\tilde{N}$  be defined by

$$\begin{cases} M := I - zL - \hat{z}U \\ N := (1 - z)L + (1 - \hat{z})U \\ \tilde{M} := I - |z||L| - |\hat{z}||U| \\ \tilde{N} := |1 - z||L| + |1 - \hat{z}||U| \end{cases} \quad (2.5)$$

where  $z$  and  $\hat{z}$  are any complex valued parameters. Then,

**LEMMA 2.2** *Under the hypothesis of Lemma 2.1 and for all complex  $z$  and  $\hat{z}$  satisfying*

$$|z|^q |\hat{z}|^s \bar{\mu}^p < 1 \quad (2.6)$$

both  $M$  and  $\tilde{M}$  are nonsingular and such that

$$0 \leq |M^{-1}| \leq \tilde{M}^{-1}. \quad (2.7)$$

*Proof* Application of Lemma 2.1 implies that, for  $z$  and  $\hat{z}$  satisfying (2.6),  $\tilde{M}$  is nonsingular and

$$\tilde{M}^{-1} \geq 0. \quad (2.8)$$

If we write  $\tilde{M} = I - \tilde{E}$ , it follows from Eq. (2.5) that  $\tilde{E} \geq 0$  so that, from Theorem 3.8 of [32],  $\rho(\tilde{E}) < 1$ . Now if  $M = I - E$ ,  $|E| \leq \tilde{E}$  so that  $\rho(E) < 1$ . Therefore, infinite expansions of both  $(I - \tilde{E})^{-1}$  and  $(I - E)^{-1}$ , in the power of  $\tilde{E}$  and  $E$  respectively, are valid, and comparing these yields  $|M^{-1}| \leq \tilde{M}^{-1}$ . ■

Using the Lemma above we now prove:

**THEOREM 2.1** *Let  $|B|$  of (2.1) be a GCO  $(s, q)$ -matrix and  $p := s + q$ . Then, for any complex  $z$  and  $\hat{z}$  satisfying*

$$(|z| + |1 - z|)^q (|\hat{z}| + |1 - \hat{z}|)^s \bar{\mu}^p < 1, \quad (2.9)$$

there holds

$$\rho(M^{-1}N) \leq \rho(\tilde{M}^{-1}\tilde{N}) < 1, \quad (2.10)$$

where  $M, N, \tilde{M}$  and  $\tilde{N}$  are as defined in (2.5).

*Proof* Using Lemma 2.2 and since  $\tilde{N} \geq 0$  it is easily seen that, for all  $z$  and  $\hat{z}$  satisfying (2.9), since (2.9) implies (2.3) and hence Lemma 2.1,

$$0 \leq |M^{-1}N| \leq \tilde{M}^{-1}\tilde{N},$$

hence, by Theorem 2.8, p. 47 of [32],

$$\rho(M^{-1}N) \leq \rho(\tilde{M}^{-1}\tilde{N}). \quad (2.11)$$

Moreover, as both  $\tilde{M}^{-1}$  and  $\tilde{N}$  are nonnegative,

$$\tilde{A} := \tilde{M} - \tilde{N} = I - (|z| + |1 - z|)|L| - (|\hat{z}| + |1 - \hat{z}|)|U| \quad (2.12)$$

is a regular splitting (cf. [32]), with (by Lemma 2.1)

$$\tilde{A}^{-1} \geq 0, \quad (2.13)$$

for all  $z$  and  $\hat{z}$  satisfying (2.9). Upon application of Theorem 3.13, p. 89 of [32], we have

$$\rho(\tilde{M}^{-1}\tilde{N}) < 1,$$

which combined with (2.11) completes the proof. ■

At this point we would like to remark that a generalized analog of the Theorem above has been obtained and used, in [33], for the determination of convergence domains for preconditioned iterative methods.

Now set

$$\mathcal{T}_r := (I - rL)^{-1}D^{-1}A = (I - rL)^{-1}(I - L - U), \quad (2.14)$$

and, recalling (1.6), observe that

$$\mathcal{L}_{r,\omega} = I - \omega\mathcal{T}_r, \quad (2.15)$$

where  $\mathcal{L}_{r,\omega}$  is the block AOR iteration matrix of (1.5).

Apparently then  $\lambda$  and  $\tau$  are respectively eigenvalues of  $\mathcal{L}_{r,\omega}$  and  $\mathcal{T}_r$ , if and only if

$$\lambda = 1 - \omega\tau, \quad (2.16)$$

whence,

$$\rho(\mathcal{L}_{r,\omega}) < 1 \quad \text{iff} \quad \tau \neq 0 \quad \text{and} \quad |1 - \omega\tau| < 1. \quad (2.17)$$

Notice that, for  $\omega > 0$ ,  $|1 - \omega\tau| < 1$  is equivalent to  $2\text{Re}(\tau) > \omega|\tau|^2$ , where  $\text{Re}(\tau)$  denotes the real part of  $\tau$ .

Assuming now that  $z$  satisfies

$$z = 1 - (1 - \hat{z})(1 - r), \quad \hat{z} \neq 1 \quad (2.18)$$

one can easily verify that

$$\begin{cases} M = (1 - \hat{z})(I - rL) \left[ I + \frac{\hat{z}}{1 - \hat{z}} \mathcal{T}_r \right] \\ N = (1 - \hat{z})(I - rL)(I - \mathcal{T}_r), \end{cases} \quad (2.19)$$

whence

$$M^{-1}N = \left[ I + \frac{\hat{z}}{1 - \hat{z}} \mathcal{T}_r \right]^{-1} (I - \mathcal{T}_r). \quad (2.20)$$

It is thus apparent that, for all  $z$  and  $\hat{z}$  satisfying (2.9) and (2.18), Theorem 2.1 implies that

$$\left| \frac{1-\tau}{1+\frac{\hat{z}}{1-\hat{z}}\tau} \right| < 1, \quad (2.21)$$

for any eigenvalue  $\tau$  of  $\mathcal{T}_r$  of (2.14). This relationship implies that  $\tau \neq 0$  and that, for  $\hat{z} < 1$ ,

$$\frac{2\operatorname{Re}(\tau)}{|\tau|^2} > 1 - \frac{\hat{z}}{1-\hat{z}}. \quad (2.22)$$

Combination of (2.17) and (2.22) yields that, for all  $z$  and  $\hat{z}$  satisfying

$$\begin{cases} (|z|+|1-z|)^q(|\hat{z}|+|1-\hat{z}|)^s \bar{\mu}^p < 1 \\ z = 1 - (1-\hat{z})(1-r) \\ \hat{z} \leq (1-\omega)/(2-\omega), \quad 0 < \omega < 2, \end{cases} \quad (2.23)$$

there holds  $\rho(L_{r,\omega}) < 1$ .

Following the analysis above we establish that:

**THEOREM 2.2** *Let  $A$  of (1.2) be an element of the matrix set  $G$  of (1.4). Then, for any  $r$  and  $\omega$  satisfying*

$$\begin{cases} 0 < \omega < 2 \\ (|1-\omega+r|+|1-r|)^q(1+|1-\omega|)^s \bar{\mu}^p < (2-\omega)^p, \end{cases} \quad (2.24)$$

where  $p := s+q$  and  $\bar{\mu} := \rho(B^A)$ , the block AOR method, applied to the matrix equation (1.1), converges (i.e.  $\rho(\mathcal{L}_{r,\omega}) < 1$ ).

*Proof* Upon setting

$$z = \frac{1-\omega+r}{2-\omega} \quad \text{and} \quad \hat{z} = \frac{1-\omega}{2-\omega},$$

observe that (2.23) holds for all  $r$  and  $\omega$  satisfying (2.24), and hence the proof follows. ■

As an immediate consequence of the above, one can easily obtain that:

**COROLLARY 2.1** *Under the hypotheses of Theorem 2.2 and for*

$$\begin{cases} 0 \leq \bar{\mu} := \rho(B^A) < 1 \\ 0 < \omega \leq 1 \quad \text{and} \quad r_1(\omega; \bar{\mu}) < r < r_2(\omega; \bar{\mu}) \\ 1 < \omega < \frac{2}{1+\bar{\mu}^{p/s}} \quad \text{and} \quad r^-(\omega; \bar{\mu}) < r < r^+(\omega; \bar{\mu}), \end{cases} \quad (2.25)$$

where

$$\begin{cases} r_1(\omega; \bar{\mu}) = \frac{\omega(1 + \bar{\mu}^{p/q}) - 2}{2\bar{\mu}^{p/q}} \\ r_2(\omega; \bar{\mu}) = \frac{2 - \omega(1 - \bar{\mu}^{p/q})}{2\bar{\mu}^{p/q}} \\ r^\pm(\omega; \bar{\mu}) = \frac{1}{2} \left\{ \omega \pm \left[ \frac{(2 - \omega)^p}{\omega^s \bar{\mu}^p} \right]^{1/q} \right\}, \end{cases} \quad (2.26)$$

the block AOR method, applied to the matrix equation (1.1), converges.

**COROLLARY 2.2** Under the hypotheses of Theorem 2.2 and for

$$\begin{cases} 0 \leq \bar{\mu} = \rho(B^A) < 1 \\ 0 < \omega < \frac{2}{1 + \bar{\mu}}, \end{cases} \quad (2.27)$$

the block SOR method, applied to the matrix equation (1.1), converges.

Concluding this section, it is worthwhile to remark that the results in Corollary 2.1 extend the convergence results found in [23].

### 3. UPPER BOUNDS OF CONVERGENCE

We begin the analysis for the determination of some upper bounds of convergence for the AOR method, by stating the following result (cf. [23]):

**THEOREM 3.1** Let  $B^A$  of (1.3) be a GCO  $(s, q)$ -matrix and  $p = s + q$ . If  $\omega \neq 0$ ,  $\lambda$  is an eigenvalue of  $\mathcal{L}_{r, \omega}$  of (1.5), with  $\lambda \neq 1 - \omega$  if  $r = 1$ , and  $\mu$  satisfies

$$(\lambda + \omega - 1)^p = (\lambda r + \omega - r)^q \omega^s \mu^p, \quad (3.1)$$

then  $\mu$  is an eigenvalue of  $B^A$ . Conversely, if  $\mu$  is an eigenvalue of  $B^A$  and  $\lambda$  satisfies (3.1), then  $\lambda$  is an eigenvalue of  $\mathcal{L}_{r, \omega}$ .

Based on the above we prove:

**THEOREM 3.2** If one of the following:

$$\begin{aligned} & a) \omega \leq 0 \quad \text{or} \quad \omega \geq 2 \\ & b) \bar{\mu} = \rho(B^A) \geq 1, \quad \text{for} \quad \omega > 0 \\ & c) (2 - \omega)^p \leq |2r - \omega|^q \omega^s \bar{\mu}^p \end{aligned} \quad (3.2)$$

holds, then

$$\sup\{\rho(\mathcal{L}_{r,\omega}^A): A \text{ is an element of } \mathcal{G}\} \geq 1, \quad (3.3)$$

where  $\mathcal{G}$  is as defined in (1.4).

*Proof* Given  $A$  in  $\mathcal{G}$ , let  $B^A$  be a GCO  $(s, q)$ -matrix. If, besides that,

a)  $\mu = 0$  is an eigenvalue of  $B^A$ , then Theorem 3.1 implies that  $\lambda := 1 - \omega$  is an eigenvalue of  $\mathcal{L}_{r,\omega}$ . Thus

$$|1 - \omega| \geq 1 \Rightarrow \rho(\mathcal{L}_{r,\omega}) \geq 1. \quad (3.4)$$

b)  $\mu^p = \bar{\mu}^p$  is an eigenvalue of  $(B^A)^p$ , with  $\bar{\mu} := \rho(B^A)$ , then Theorem 3.1 implies that any  $\lambda$  which satisfies

$$P(\lambda; r, \omega, \bar{\mu}) = 0, \quad (3.5)$$

where  $P(\lambda; r, \omega, \bar{\mu}) := (\lambda + \omega - 1)^p - (\lambda r + \omega - r)^q \omega^s \bar{\mu}^p$ , is an eigenvalue of  $\mathcal{L}_{r,\omega}$ . Observe now that, for  $\bar{\mu} \geq 1$  and  $\omega > 0$ ,

$$P(1; r, \omega, \bar{\mu}) = \omega^p (1 - \bar{\mu}^p) \leq 0, \quad (3.6)$$

which combined with the fact that

$$P(\lambda; r, \omega, \bar{\mu}) > 0, \quad \text{for } \lambda, p \text{ sufficiently large}, \quad (3.7)$$

implies that there exists  $\lambda^* \geq 1$  such that  $P(\lambda^*; \omega, r, \bar{\mu}) = 0$ . Thus, as (3.5) implies that  $\lambda^*$  is an eigenvalue of  $\mathcal{L}_{r,\omega}$ , we have that

$$\bar{\mu} \geq 1 \quad \text{and} \quad \omega > 0 \Rightarrow \rho(\mathcal{L}_{r,\omega}) \geq 1. \quad (3.8)$$

c)  $\mu^p = (-1)^s \bar{\mu}^p$  is an eigenvalue of  $(B^A)^p$ , with  $\bar{\mu} := \rho(B^A)$ , and  $q$  even, Theorem 3.1 implies that if  $v$  satisfies

$$\bar{P}(v; r, \omega, \bar{\mu}) = 0, \quad (3.9)$$

where  $\bar{P}(v; r, \omega, \bar{\mu}) := (v - \omega + 1)^p - |vr - \omega + r|^q \omega^s \bar{\mu}^p$ , and  $\lambda := -v$  then  $\lambda$  is an eigenvalue of  $\mathcal{L}_{r,\omega}$ . By a similar argument as in (b) above, one can easily verify that

$$(2 - \omega)^p \leq |2r - \omega|^q \omega^s \bar{\mu}^p \Rightarrow \rho(\mathcal{L}_{r,\omega}) \geq 1. \quad (3.10)$$

Combination of (3.4), (3.8) and (3.10) completes the proof. ■

At this point it is worthwhile to remark that the relation (3.2c) can equivalently be expressed as

$$r \leq r^-(\omega; \bar{\mu}) \quad \text{and} \quad r \geq r^+(\omega; \bar{\mu}) \quad (3.11)$$

where  $r^\pm(\omega; \bar{\mu})$  are as defined in (2.26).

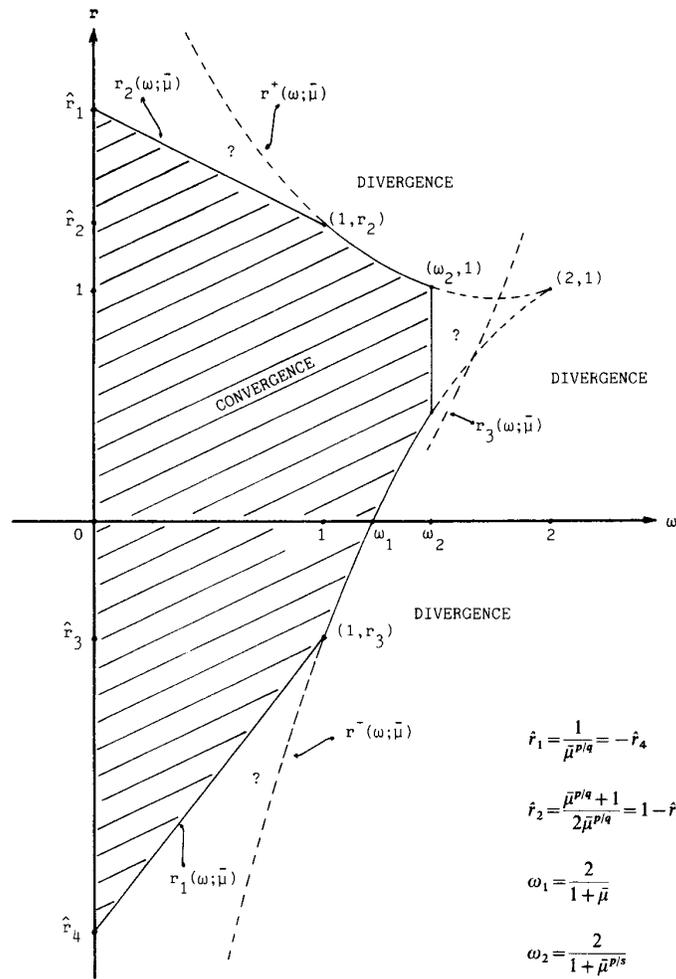


Figure 1 Regions of convergence and divergence of the block AOR method, as it pertains to GCO  $(s, q)$  matrices.

4. CONCLUSION AND ILLUSTRATIONS

For clarity, the results from Corollary 2.1 and Theorem 3.2 are graphically represented in Figure 1. Observe that for

$$1 \leq \omega < \frac{2}{1 + \mu^{p/s}}$$

the regions of convergence and divergence *exactly* complement each other.

Concluding, and as a first step in answering the question of whether convergence or divergence or both occur in the regions marked by “?” of Figure 1, the following two remarks are included.

*Remark 1* Recalling the curves defined in (2.26), one can easily verify that, for  $s$  fixed,

$$\lim_{q \rightarrow \infty} \{r^+(\omega; \bar{\mu})\} = \lim_{q \rightarrow \infty} \{r_2(\omega; \bar{\mu})\} = \frac{2 - \omega(1 - \bar{\mu})}{2\bar{\mu}}$$

$$\lim_{q \rightarrow \infty} \{r^-(\omega; \bar{\mu})\} = \lim_{q \rightarrow \infty} \{r_1(\omega; \bar{\mu})\} = \frac{\omega(1 + \bar{\mu}) - 2}{2\bar{\mu}}.$$

Therefore, for  $q$  sufficiently large and for  $0 < \omega \leq 1$ , the curves which separate the domains of convergence and divergence “exactly” approach each other.

*Remark 2* Utilizing the constant term of the characteristic polynomial associated with  $\mathcal{L}_{r,\omega}$  of (1.5), we established (the proof is omitted for brevity) that, for

$$\omega \geq \frac{2}{1 + \bar{\mu}^{p/s}} \quad \text{and} \quad r \leq \omega - \left[ \frac{1 - (\omega - 1)^p}{\omega^s \bar{\mu}^p} \right]^{1/q} =: r_3(\omega; \bar{\mu})$$

there holds

$$\rho(\mathcal{L}_{r,\omega}^A) \geq 1, \text{ for any GCO } (s, q)\text{-matrix } A.$$

Finally, we would like to remark that a similar analysis has been used in [34] for the derivation of exact convergence regions of the SSOR method, as it pertains to  $H$ -matrices.

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